Nonparametric frequency response function (FRF) estimation [H04Q7]

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Introduction

- discuss nonparametric estimation of the frequency response function (FRF) of linear time invariant single-input single-output systems.
- broadband periodic excitation is considered
- input and output are perturbed by stochastic errors

**Exact data:**

\[ G(k) = \frac{Y(k)}{U(k)} \]

**Measured Fourier Coefficients:**

\[ U_m(k) = U(k) + N_u(k) \]
\[ Y_m(k) = Y(k) + N_y(k) \]
• Properties of noise:

\[
E\{N_{r_u}(k)N_{r_u}(l)\} = E\{N_{i_u}(k)N_{i_u}(l)\} = \delta(k,l)\sigma_u^2(k)
\]
\[
E\{N_{r_u}(k)N_{i_u}(l)\} = 0
\]
\[
E\{N_{r_y}(k)N_{r_y}(l)\} = E\{N_{i_y}(k)N_{i_y}(l)\} = \delta(k,l)\sigma_y^2(k)
\]
\[
E\{N_{r_y}(k)N_{i_y}(l)\} = 0
\]
\[
E\{N_{r_u}(k)N_{r_y}(l)\} = E\{N_{i_y}(k)N_{i_u}(l)\} = \delta(k,l)\rho_r(k)
\]
\[
E\{N_{r_u}(k)N_{i_y}(l)\} = -E\{N_{i_u}(k)N_{r_y}(l)\} = \delta(k,l)\rho_i(k)
\]

• The empirical transfer function estimate

\[
G_m(k) = \frac{Y_m(k)}{U_m(k)} = G(k)\frac{1 + \frac{N_y(k)}{Y(k)}}{1 + \frac{N_u(k)}{U(k)}}
\]
- To reduce the effect of stochastic error on the FRF estimate: use averaging techniques: consider
  - a sequence of spectral input-output data \( \{ U_{m_i}(k), Y_{m_i}(k) \} \),
    \( i = 1, 2, \ldots, M \)
  - obtained by applying a DFT on \( M \) nonoverlapping time records
    \( \{ u_m(t_i + nT_s), y_m(t_i + nT_s) \} \), with \( i = 1, 2, \ldots, M \), \( n = 0, 1, \ldots, N - 1 \),
    \( t_{i+1} \geq t_i + NT_s \)
  - if \( t_{i+1} = t_i + NT_s \): the measurements are synchronized with the excitation
Different nonparametric estimators: Classical estimators

- the classical $G_1$ and $G_2$ estimators are defined as:

$$
\hat{G}_1(k) = \frac{1}{M} \sum_{i=1}^{M} U_{m_i}^*(k) Y_{m_i}(k) \\
\hat{G}_2(k) = \frac{1}{M} \sum_{i=1}^{M} Y_{m_i}^*(k) Y_{m_i}(k)
$$

- or

$$
\hat{G}_1(k) = \frac{\hat{G}_{U_m Y_m}(k)}{\hat{G}_{U_m U_m}(k)} \\
\hat{G}_2(k) = \frac{\hat{G}_{Y_m Y_m}(k)}{\hat{G}_{Y_m U_m}(k)}
$$
where an estimate of the power spectrum $\hat{G}_{VW}$ is given by:

$$\hat{G}_{VW}(k) = \text{ave}(V^*(k)W(k))$$

and

$$\text{ave}(W(k)) = \frac{1}{M} \sum_{i=1}^{M} W_i(k)$$
• The $G_1$ estimator corresponds to the least squares estimate of $G$.

$$\hat{G}_1(k) = \arg \min_{G(k)} \frac{1}{M} \sum_{i=1}^{M} |G(k)U_{mi}(k) - Y_{mi}(k)|^2$$

• Matrix formulation of LSE : overdetermined set of equations

$$U_m(k)G(k) = Y_m(k)$$

with

$$U_m(k) = \begin{bmatrix} U_{m1}(k) & U_{m2}(k) & \cdots & U_{mM}(k) \end{bmatrix}^T$$

$$Y_m(k) = \begin{bmatrix} Y_{m1}(k) & Y_{m2}(k) & \cdots & Y_{mM}(k) \end{bmatrix}^T$$

$$\hat{G}_{LS}(k) = \left(U_m^H(k)U_m(k)\right)^{-1} U_m^H(k)Y_m(k)$$
The $G_2$ estimator corresponds to the *least squares estimate of $G^{-1}$.*

$$\hat{G}_2(k) = \frac{1}{\arg \min_{G^{-1}}(k) \sum_{i=1}^{M} |U_{m_i}(k) - G^{-1}(k)Y_{m_i}(k)|^2}$$
Different nonparametric estimators: Nonlinear estimators

**Analytic estimators**

- Based on nonlinear means of the ETFE:
  \[ G_{mi}(k) = \frac{Y_{mi}(k)}{U_{mi}(k)} \quad i = 1, \ldots, M \]

- General formulation of a nonlinear mean:
  \[ \hat{G}_g(k) = g^{-1}(\text{ave}(g(G_{m}(k)))) \]

- Some interesting \( g \)-functions are:
  - *arithmetical mean*: \( g(x) = x \), which yields the estimator:
    \[ \hat{G}_{ari}(k) = \text{ave}(G_{m}(k)); \]
  - *harmonic mean*: \( g(x) = 1/x \), which yields the estimator:
    \[ \hat{G}_{har}(k) = 1/\text{ave}(1/G_{m}(k)); \]
- geometric mean: \( g(x) = \ln(x) \), which yields the estimator:
\[
\hat{G}_{ln}(k) = \exp(\text{ave}(\ln(G_m(k))));
\]
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**Synchronization**

- These estimators as well as $G_1$ and $G_2$ can be applied even if it is impossible to synchronize the measurements.

- I.e. if the sequences of spectral input-output data $\{U_{m\ i}(k), Y_{m\ i}(k)\}$, $i = 1, 2, \ldots, M$ are obtained from $M$ non-overlapping time records $\{u_m(t_i + nT_s), y_m(t_i + nT_s)\}$, with $t_{i+1} \neq t_i + NT_s$ ($i = 1, 2, \ldots, M$, $n = 1, 2, \ldots, N - 1$).

- If synchronization is possible: use an unbiased *errors-in-variables* estimator.

- Synchronization of the generator and the data acquisition yields:
  - $E\{U_{m\ i}(k)\} = U(k)$, for $i = 1, \ldots, M$, and for $k = 1, \ldots, N$
  - $E\{Y_{m\ i}(k)\} = Y(k)$, for $i = 1, \ldots, M$, and for $k = 1, \ldots, N$

- The errors-in-variables estimator $G_{EV}$:

$$
\hat{G}_{EV}(k) = \frac{\text{ave}(Y_m(k))}{\text{ave}(U_m(k))} = \frac{1}{M} \sum_{i=1}^{M} Y_{m\ i}(k) = \frac{1}{M} \sum_{i=1}^{M} U_{m\ i}(k)
$$
• This is the maximum likelihood estimator (MLE) if the input and output perturbations are complex normal distributed (even if they are mutually correlated).
**MLE of the FRF**

- Gaussian p.d.f. of the errors, for each spectral line $k$,

- constraint between the input and output Fourier coefficient

\[
Y(k) = G(k)U(k)
\]

- include constraints $\lambda_r$ and $\lambda_i$

- resulting loss function

\[
V(G, Y, U, \lambda_r, \lambda_i) = \sum_{i=1}^{M} (Z_{mi} - Z)^H C^{-1} (Z_{mi} - Z) + \lambda_r e_r + \lambda_i e_i
\]

- with measurements

\[
Z_{mi} = \begin{bmatrix} U_{mi}(k) \\ Y_{mi}(k) \end{bmatrix}
\]
and

\[
Z = \begin{bmatrix}
U(k) \\
Y(k)
\end{bmatrix}
\]

- the covariance matrix for all measurements \(i\):

\[
C = E \left\{ (Z_{mi} - E\{Z_{mi}\})(Z_{mi} - E\{Z_{mi}\})^H \right\} \\
= 2 \begin{bmatrix}
\sigma_u^2 & \rho \\
\rho^* & \sigma_y^2
\end{bmatrix}
\]
• the derivative of the loss function with respect to the unknown Fourier coefficients $Z$ yields:

$$-2C^{-1}\left(\sum_{i=1}^{M} Z_{mi} - Z\right) + B^H \lambda = 0$$

with

$$B = \begin{bmatrix} G & -1 \end{bmatrix}$$

$$\lambda = \lambda_r + j\lambda_i$$

• the derivative with respect to $G$ yields: $\lambda = 0$.

• Including $\lambda = 0$ yields:

$$\hat{U}_{ML}(k) = \frac{1}{M} \sum_{i=1}^{M} U_{mi}(k)$$

$$\hat{Y}_{ML}(k) = \frac{1}{M} \sum_{i=1}^{M} Y_{mi}(k)$$
• Invariance property of the MLE yields:

\[ \hat{G}_{EV} = \hat{G}_{ML}(k) = \frac{\sum_{i=1}^{M} Y_{mi}(k)}{\sum_{i=1}^{M} U_{mi}(k)} \]

• The Cramér-Rao bound for this MLE is given by:

\[ \sigma_{CR}^2 = 2 \frac{|G|^2}{M} \left( \frac{\sigma_u^2}{U^2} + \frac{\sigma_y^2}{Y^2} - 2 \Re \left( \frac{\rho}{U*Y} \right) \right) \]

• As the ML estimator is asymptotically efficient, the p.d.f. of \( \hat{G}_{EV} \) converges for \( M \to \infty \) to \( N(G, \sigma_{CR}^2) \)
Verification of model accuracy

• Comparing the maximum likelihood FRF estimate with the frequency response of the model is one of the model validation tests.

• The distance between these FRF’s can be compared with the confidence level on the maximum likelihood FRF estimate (Cramér-Rao bound)

\[
|\hat{G}_{EV}(k) - G(\Omega_k, \hat{\theta})|^2 = (\hat{G}_{EVr}(k) - G_r(\Omega_k, \hat{\theta}))^2 + (\hat{G}_{EVi}(k) - G_i(\Omega_k, \hat{\theta}))^2,
\]

• this distance is a random variable with a $\chi^2(2)$ distribution.

•

\[
|\hat{G}_{EV}(k) - G(\Omega_k, \hat{\theta})|^2 < N_\alpha \frac{\sigma_{CR}^2(k)}{2}
\]
Properties of the FRF estimators

- The mean square error (MSE) of an estimator \( \hat{G}(k) \):

\[
MSE(\hat{G}(k)) = E\{|\hat{G}(k) - G(k)|^2\} = b^2(\hat{G}(k)) + \text{var}(\hat{G}(k))
\]

- Most attention will be given to the analysis of the bias (systematic error):

\[
b(\hat{G}(k)) = E\{\hat{G}(k)\} - G(k)
\]

- Since the nonsystematic error can always be reduced by averaging:

\[
\text{var}(\hat{G}(k)) = E\{|\hat{G}(k) - E\{\hat{G}(k)\}|^2\}
\]

- The bias in decibel is defined as:

\[
b_{dB}(\hat{G}(k)) = 20 \log_{10}(|E\{\hat{G}(k)\} - G(k)|)
\]
The difference between the mathematical expectation of the FRF estimate and the exact value in decibel is denoted by:

$$\delta_{dB}(\hat{G}(k)) = 20 \log_{10}(|E\{\hat{G}(k)\}|) - 20 \log_{10}(|G(k)|)$$
Expressions for the bias when the input-output errors are complex normal distributed and mutually uncorrelated

- First define the signal-to-noise ratio (SNR) of $U_m(k)$ and $Y_m(k)$:

\[
SNR(U_m(k)) = \frac{U(k)U^*(k)}{2\sigma^2_u(k)}
\]

\[
SNR(Y_m(k)) = \frac{Y(k)Y^*(k)}{2\sigma^2_y(k)}
\]
• It can be shown that:
  – The phase of all the presented estimators is unbiased for Gaussian errors.

\[
\begin{align*}
\delta_{dB}(\hat{G}_{EV}) &= 0 \\
\delta_{dB}(\hat{G}_{ari}) &= 20 \log_{10} (1 - \exp(-SNR(U_m(k)))) \\
\delta_{dB}(\hat{G}_{har}) &= -20 \log_{10} (1 - \exp(-SNR(Y_m(k)))) \\
\delta_{dB}(\hat{G}_{log}) &= \frac{10}{\ln(10)} (Ei(-SNR(U_m(k))) - Ei(-SNR(Y_m(k)))) \\
\delta_{dB}(\hat{G}_1) &= -20 \log_{10} \left(1 - \frac{1}{SNR(U_m(k))}\right) \\
\delta_{dB}(\hat{G}_2) &= 20 \log_{10} \left(1 - \frac{1}{SNR(Y_m(k))}\right)
\end{align*}
\]

where \(Ei\) represents the exponential-integral function:

\[
Ei(-x) = - \int_x^\infty \frac{\exp(-t)}{t} dt
\]
| SNR   | $|\delta_{dB}(\hat{G}_{log})|$ | $|\delta_{dB}(\hat{G}_{har})| \& |\delta_{dB}(\hat{G}_{ari})|$ | $|\delta_{dB}(\hat{G}_{1})| \& |\delta_{dB}(\hat{G}_{2})|$ |
|-------|------------------------------|----------------------------------|----------------------------------|
| $> 6dB$ | $< 17 mdB$                  | $< 0.16 dB$                      | $< 1.9 dB$                      |
| $> 10dB$ | $< 18 \mu dB$              | $< 39 mdB$                      | $< 0.83 dB$                     |
| $> 20dB$ | $< 2e - 45 dB$         | $< 3e - 43 dB$                  | $< 86 mdB$                     |

Tabel 1: Maximum difference $\delta_{dB}$ for uncorrelated Gaussian input-output errors
Bias when the input-output errors are complex normal distributed and mutually correlated

- the magnitude as well as the phase obtained with the classical estimators (e.g., $G_1$, and $G_2$) are biased and depends on the correlation factors.

- the bias of the analytic estimators is a function of the fourth-order moments of the frequency domain noise.

- For normal distributed errors, the bias of the $G_{log}$ estimator is independent of the correlation factor $\rho$ and the phase is unbiased.
**Practical considerations: Phase corrections**

- The phase estimates for the $G_{log}$ estimator are in practical applications biased.
- This is due to the “cut” of the complex logarithmic function along the negative real axis.
- **Example:**
  - two observations: $A \exp(j(180 \text{ deg} - \alpha))$ and $A \exp(j(180 \text{ deg} + \alpha))$.
  - the geometric mean becomes:

$$\hat{G}_{log} = \exp (\ln(A) + j [(180 \text{ deg} - \alpha) + (-180 \text{ deg} + \alpha)]/2) = A$$
• **Remedy:**
  
  – stay away from this “cut” by rotating the data
  – angle of rotation \( \varphi(k) \): the phase of the cross-power spectrum \( \hat{G}_{U_m,Y_m}(k) \).
  – rotate all the \( M \) measurements at spectral line \( k \) over angle \(-\varphi(k)\) radians
  – then apply the \( G_{\text{log}} \) estimator
  – rotate the estimate over \( \varphi(k) \) radians
Practical considerations: Outliers

- outliers: occasional large measurement errors

- The $G_{ari}$ is robust input outliers.
  
  $$\hat{G}_{ari}(k) = \frac{1}{M} \sum_{i=1}^{M} \frac{Y_{mi}(k)}{U_{mi}(k)}$$

- The $G_{har}$ is robust output outliers.

- If output \textbf{and} input outliers the $G_{log}$ estimator is to be preferred (due to log function which will considerably attenuate the effect of outliers).
Summary

- Different FRF estimators
- Which estimator is preferable in which situation
- What is required to apply the MLE?
- Influence of outliers and phase correction