Experimental Flexible Beam Tip Tracking Control with a Truncated Series Approximation to Uncancelable Inverse Dynamics

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Abstract—A feedforward design methodology to compensate unstable zeros in linear discrete-time systems with tracking objectives is reviewed. An experimental study for end-point tracking of a flexible beam was performed to validate the effectiveness of the proposed scheme. Results are compared with previous results

The methodology exploits the fact that the noncausal expansion of unstable inverse dynamics is convergent in the region of the complex plane encompassing the unit circle. An approximation to the unstable inverse dynamics that can be implemented follows, by truncating the series and utilizing the necessary preview information. Right-half plane unstable zeros near the unit circle can significantly reduce bandwidth. For such zeros, the series methodology is shown experimentally to yield better performance than ZPET, since both gain as well as phase are compensated.

I. INTRODUCTION

Many mechanical systems require accurate high speed tracking capabilities. This can usually be realized with the application of feedforward techniques in conjunction with a feedback design. With knowledge of the system dynamics a feedforward scheme can anticipate the effect of the closed-loop dynamics and adjust the reference trajectory accordingly. This anticipation amounts to inversion of the dynamics in a specified bandwidth. Feedforward plant inversion has a number of issues which must be addressed such as causality, high gain, robustness, and unstable inverse dynamics. This paper addresses the issue of unstable inverse dynamics. It is not uncommon for unstable inverse dynamics to occur in discrete-time transfer functions that represent mechanical systems. In fact, the zero-order-hold equivalent of a continuous-time plant with relative degree greater than two always yields left-half plane unstable zeros for sufficiently fast sampling rates [1]. Right-half plane unstable zeros are encountered in practice less frequently. They are usually observed, however, in system models that are dispersive and have noncolocated sensors and actuators [10]. The flexible beam has been a common paradigm to study control issues for such systems. The inherent difficulties of robust feedback control of such systems are well understood, and details can be found in [2], [13], [15], and [10]. From the latter works, it is clear that a robust solution, with regards to stability, is the design of a colocated control coupled with feedforward compensation.

A number of feedforward techniques have been developed to minimize the effect of unstable zeros on tracking performance. Among these is the zero-phase error tracking scheme (ZPET) developed by Tomizuka [17], [16], which is a noncausal feedforward compensation based on partial plant inversion. All poles and specified stable zeros in a stable or previously stabilized system are canceled. The essence of the scheme is to include the inverse of the remaining zeros as zeros in the feedforward compensation block. This has the effect of completely eliminating any phase error. The controller is then adjusted such that the steady-state gain from input to output is unity. Note that ZPET can be applied to cancel the phase error caused by zeros on the unit circle, and stable zeros that otherwise are not chosen to be canceled. These include stable left-half plane zeros which, when inverted, may yield an undesirable oscillatory mode in the reference signal.

It is possible that after the application of ZPET, the overall bandwidth may be insufficient. To compensate for the gain error of the plant and that induced by the ZPET controller, Haack and Tomizuka [5] developed a systematic means of including additional zeros to reduce the gain error and preserve the zero-phase error characteristics. The zero locations are specified by minimizing an expression for two axis circular contouring error and is applicable for right-half plane zeros (though in principle it can be extended to left-half plane zeros).

A classification of discrete-time unstable zeros and two feedforward compensation schemes for precision tracking control have been presented by Menq and Xia [9]. The classification and design procedure is, in general, very useful. The design procedure can be numerically cumbersome, however, and is subject to certain approximations from which the final result may not be apparent.

Both [6], [7] have presented a novel class of feedforward controllers for both continuous-time and discrete-time systems that in the absence of model uncertainty, noise, and external disturbances, assures perfect tracking for a class of reference signals. The limitations are, of course, on the class of signals tracked. Also, the scheme is not robust to signals that may slightly deviate from those specified.

In this paper, a discrete-time unstable zero compensation design methodology that is simple and effective is described and then experimentally applied. The method compensates for gain as well as phase simultaneously. Although the method is

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approximate, the approximation errors can be made arbitrarily small. The approximation has been previously utilized for minimum variance control and model reference control [12], [18]. It has also been utilized in a preview control context [11], but design guidelines, frequency weighting [4], and experimental results have not been presented. Simple rules are described to determine the required order of the filter from frequency-domain tracking design specifications. For shaping the gain versus frequency characteristics of a zero-phase filter, a notch filter for example, the methodology allows one to utilize the gain shaping properties of unstable as well as stable poles. One then replaces the unstable poles with stable approximations for implementation purposes.

This paper is organized into four additional sections. In Section II, the design procedure is described. In Section III, extensions to the compensation of stable and marginally stable zeros and to the design of zero-phase filters are discussed. In Section IV, experimental results in positioning the tip of a highly flexible beam are described. A comparison with ZPET augmented with a zero-phase low-pass filter is included. Concluding remarks are given in Section V.

II. DESIGN PROCEDURE

It is the aim of this section to review a general design methodology to compensate gain and phase distortions caused by unstable zeros in linear time-invariant, discrete-time systems, i.e., those zeros that occur outside the unit circle. Details of the methodology can be found in [4]. Suppose a known discrete-time single-input single-output system is under control. The closed-loop system consisting of a plant and a feedback controller can be described as

$$G(z^{-1}) = \frac{z^{-d}B^+(z^{-1})B^-(z^{-1})}{A(z^{-1})}$$  

(1)

where $d$ is the number of pure delays, $B^+(z^{-1})$ represents the polynomial of stable zeros, $B^-(z^{-1})$ represents the polynomial of unstable zeros, and the roots of $A(z^{-1})$ are the system poles. By use of a feedforward compensator and future reference information, one can cancel by plant inversion the delay, the poles, and the stable zeros. The remaining dynamics are those of the uncancellable zeros. In the following development, we will assume that the term uncancellable zeros will refer only to unstable zeros. The case of zeros on the unit circle is discussed in Section III.

Suppose that a system has only one real unstable zero at $\alpha$ and no zeros on the unit circle. $\alpha$ can represent either a left half or right-half plane zero. To cancel the zero at $\alpha$ we cannot include $\frac{1}{z-\alpha}$ in the feedforward controller with the interpretation that the expansion in negative powers of $z$ represents the impulse response. This is clearly unstable. By Taylor series expansion about the origin we have

$$\frac{1}{z-\alpha} = z^{-1} + \alpha z^{-2} + \alpha^2 z^{-3} + \alpha^3 z^{-4} \ldots \quad |z| > \alpha$$

$$\frac{1}{z+\alpha} = \frac{1}{\alpha} - \frac{1}{\alpha} z^{-1} - \frac{1}{\alpha^2} z^{-2} - \frac{1}{\alpha^3} z^{-3} \ldots \quad |z| < \alpha.$$  

(2)

$$\frac{1}{z^2 + \gamma z + \beta} = \frac{1}{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_n z^{-n}}.$$  

(3)

Since $|\alpha| > 1$, the infinite sequence, (3), is convergent in the region that includes the unit circle. The product of the four term truncated series from (3) and the plant zero yields

$$\frac{-1}{\alpha} + \frac{1}{\alpha^2} z^{-1} - \frac{1}{\alpha^3} z^{-2} - \frac{1}{\alpha^4} z^{-3} \ldots = 1 - \frac{z^4}{\alpha^4}.$$  

(4)

The frequency response of $1 - \frac{z^4}{\alpha^4}$ can be represented geometrically as a phasor with head at one and a tail revolving about a circle centered at the origin of radius $\frac{1}{\alpha}$. The radius of the circle decreases geometrically with the number of terms taken in the series. These observations are summarized in the following theorem.

Theorem 2.1: For a real unstable zero at $\alpha$ (root of $z-\alpha=0$), and with $n$ terms taken from the series expansion, the input–output transfer function product will be

$$1 - \frac{z^n}{\alpha^n}.$$  

(5)

The gain shall lie within a band

$$1 \pm \frac{1}{\alpha^n},$$  

(6)

and the maximum value of phase lead and lag, $\theta_{\text{max}}$, is given by

$$\theta_{\text{max}} = \arcsin\left(\frac{1}{\alpha^n}\right).$$  

(7)

Proof: See [4]

When zeros occur as complex conjugate pairs it is not at first clear how many terms, $n$, one should take. It is shown in [4] that to minimize the least square error between the inverse and the stable series approximation, one should select the terms in the expansion in the order of largest magnitude to least magnitude. If the number of terms are limited, one can also introduce frequency weighting to obtain coefficients that yield an adequate response in the bandwidth of interest.

To obtain the coefficients of the feedforward compensator for complex unstable zeros, suppose the complex pair is given by the roots of the equation

$$z^2 + \gamma z + \beta = 0.$$  

(8)

One must determine the expansion in positive powers of $z$ of

$$\frac{1}{z^2 + \gamma z + \beta}.$$  

(9)

Combinations of any number of complex conjugate pairs of unstable zeros and real zeros may occur. A general algorithm to determine the noncausal expansion is stated in the following theorem.

Theorem 2.2: Suppose all unstable zeros are represented by the $n$th order polynomial $B(z^{-1})^- = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_n z^{-n}$. It is assumed without loss of generality that $b_0 \neq 0$. The coefficients of the noncausal expansion of $\frac{1}{B(z^{-1})-}$ are simply given by the impulse response of the stable system

$$\frac{1}{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_n z^{-n}}.$$  

(10)
The impulse response of\((10)\) can be readily obtained manually or by simulation. The proof is straightforward.

**Proof:** We wish to obtain the noncausal response or in other words the expansion of

\[
\frac{1}{b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n}
\]

in positive powers of\(z\). By the simple substitution of\(z^{-1}\) for all instances of\(z\) and noting that all roots originally lie outside the unit circle, the transfer function is transformed into the conventional notation of a stable system. Since the impulse response is the stable expansion of\((10)\) in negative powers of\(z\), upon remaking the substitution, the terms are simply the coefficients of the expansion in positive powers of\(z\). The latter is precisely what we wish to compute.

**Proof:** In summary, arbitrarily good approximation to the unstable inverse dynamics can be obtained by first inverting the unstable dynamics and then expanding the inverse in positive powers of\(z\). The number of terms taken determines the degree of approximation. The closer the zeros are to the unit circle, the more terms necessary to satisfy a given tracking specifications can be met. If tracking specifications are still not met, however, then a number of feedforward schemes can be used to extend the system bandwidth [5], [9]. The feedforward series approach can also be used to extend the system bandwidth in a very simple manner while allowing direct specification and approximate achievement of a desired frequency response. Suppose there is a closed-loop system under feedback control \(G(z^{-1})\). Also based on engineering judgment, we define a desired transfer function, \(G_{des}(z^{-1})\), that we wish to achieve with the addition of feedforward compensation. The desired transfer function, \(G_{des}(z^{-1})\), is constrained to be a zero at \(-1\) (or any other zero that one has decided not to cancel) must be included. One then simply formulates the feedforward compensator as \(G_{des}(z^{-1})\). All unstable poles in the latter product are then replaced by their series approximation.

In fact, as the examples in Sections III-A and III-B will illustrate, the desired transfer function, \(G_{des}(z^{-1})\), can be specified as zero phase and noncausal if future reference trajectory information is available. If it is zero phase then for each stable pole, there must be a corresponding inverse.

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An approximate noncausal expansion based on the first eight terms of the impulse response is

\[ 1 + 1.1619z + 0.65411z^2 + 0.08925z^3 - 0.1914z^4 - 0.1943z^5 - 0.0803z^6 + 0.01554z^7 + 0.04718z^8. \]

(15)

Figs. 3 and 4 give the frequency response of the approximate zero-phase error-modified Butterworth filter. Note that the modified filter does not exhibit exact zero phase characteristics because of the truncated series approximation. It retains almost zero-phase characteristics, however, especially within the bandwidth of the filter. The expected bandwidth is achieved and the maximum phase error is about one degree.

**B. Zero-Phase Error Notch Filter**

This section describes a discrete-time design of a noncausal zero-phase notch filter. Generally, if one simply cancels the resonant poles of the system, extremely high gain at high frequencies results. A high-order low-pass filter is necessary to attenuate this effect unless extreme care has been taken so that the reference trajectory possesses small enough energy in the highly-amplified region. The approach taken here also will sometimes rely on a high-order filter. The terms of the filter are dependent, however, on only one or at most two parameters that can be easily tuned experimentally.

Take, for example, a system with a lightly-damped pole pair

\[ \frac{0.55}{z^2 - 1.26z + 0.81}. \]

(16)

The notch precompensator will contain the complex zero pair corresponding to the system poles. To ensure that the feedforward gain is not excessive at high frequencies, another pair of poles is placed near the zeros but is more damped. In this example, the canceled poles are replaced with poles of equal frequency but with a damping coefficient of 0.536. The new pole pair is 0.425±0.425j. The precompensator becomes

\[ \frac{0.51125(z^2 - 1.26z + 0.81)}{(z^2 - 0.85z + 0.36125)0.55}. \]

(17)

To maintain zero-phase characteristics the above filter is augmented by the complex conjugate pair

\[ \frac{0.51125z^2}{0.36125z^2 - 0.85z + 1}. \]

(18)

Since the poles of (18) are unstable, they must be replaced by a noncausal truncated series approximation. The following seven term series approximation to (18) is

\[ 0.0074z^9 + 0.0087z^8 - 0.024z^6 - 0.057z^5 - 0.0667z^4 + 0.185z^2 + 0.435z^1 + 0.51125. \]

(19)
The response following a unit step command with no precompensation is shown in Fig. 5. The response following precompensation with the filter in (20) is shown in Fig. 6. Lastly Fig. 7 shows the post processed reference command.

The resulting precompensator becomes

\[
\left\{ 0.0074z^{-3} + 0.0087z^{-6} - 0.024z^{-9} - 0.057z^{-12} - 0.0667z^{-15} + 0.185z^{-18} + 0.435z^{-21} + 0.51125 \right\} 
\times 
\left\{ 0.56z^{-4} - 1.26z^{-7} + 0.81 \right\}.
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Beam-tip position measurement was accomplished with a high speed Kodak SP2000 video motion analysis system. Recordings were made at a rate of 200 frames per second. A counter, timer, and recticle lines were features that allowed precise position recording.

B. System Model

The development of the following linear model for in-plane motion of a uniform flexible beam can be found in [3],[14]. We state only the essential results here. In [3] a set of linearized equations are derived subject to the assumptions of small deflections and negligible deformations due to shear. In the equations are derived subject to the assumptions of small motion of a uniform flexible beam can be found in [3],[14]. We included in the experimentally identified model. Also, aside from the gain, (21) and (22) only differ in the numerators. As confirmation, in [9], the zeros of a distributed system are the modes of the system while constrained between the actuator and sensor. Since the actuator and sensor are colocated for the transfer function from torque to hub position, the zeros are precisely the modes of the fixed axis cantilevered beam (not including the zero introduced by the PD control). The measured value was found to be 1.76 Hz. From swept sine results the first zero is about 1.78 Hz. Both measured values are in agreement within 1.1%.

Noting (21) the complex zeros are predicted to reside at the roots of the equation

$$\frac{-a^2t}{a^2}(s^2+1) + 1 = 0 \quad (25)$$

for n = 1, 2, 3, . . . , where an are the positive roots of 1 + cos(an) cosh(an) = 0. From (22), the transfer function from motor voltage to beam tip position zeros have migrated from the imaginary axis to the real axis and are now at the roots of the equation

$$\frac{-a^4}{4e^4}(s^2+1) + 1 = 0 \quad (26)$$

for n = 1, 2, 3, . . . , and bn are the positive roots of the equation tan(bn) + tanh(bn) + 2(2)bn = 0. Since the first three complex zeros were identified to be at about 1.77 Hz, 11.007 Hz, and 30.217 Hz. The errors with reference to the measured values from frequency response data are about 1% and 4% respectively. With an estimate of \( a_1 \) and using (22), estimates of the first three real zeros were computed. The resulting transfer function is (where s is now in radians/second) shown in (27) at bottom of the page.

To obtain a discrete-time zero-order-hold equivalent to (27) a sampling rate of 10ms was judged reasonable since the reference commands were slews of 130 and 200 milliseconds. The discrete-time zero-order-hold equivalent is shown in (28), at the bottom of the page. It is the system model, (28), on which the ZPET and series feedforward designs are based.
D. Experimental and Simulation Results and Comparisons

Two sets of three experiments were conducted. Each was a fast one radian slew reference trajectory of either a 130 or 200 millisecond duration. The maximum velocity of each reference trajectory is about 10.5 and \( 6.5 \) \( \text{rad/sec} \), respectively. Each trajectory consists of a segment of a sine function from \( -\pi \) to \( \pi \) radians. Experiments with no feedforward control, truncated series control, and ZPET with an added lowpass zero-phase filter were performed. The low-pass zero-phase filter utilized was the modified zero-phase Butterworth filter described in Section III-A.

For the unstable zeros at \(-5.834, 5.92, \) and \(1.407\), three, three, and eight term truncated series approximations were used. According to (6), this results in gain variations of about \( \frac{1}{100} \), \( \frac{1}{100} \), and \( \frac{1}{100} \), respectively, about unity for each approximate cancellation. The feedforward control transfer function is for the series case as shown in (29) at the bottom of the page. For the ZPET trials, the feedforward control transfer function is [16], shown in (30) at the bottom of the page. Though not shown, for the ZPET trials, the zero-phase low-pass Butterworth filter derived in Section III-A also preprocessed the reference trajectory.

Figs. 10, 11, and 12 display the 200 ms reference command and the simulated plant response with the series, ZPET with zero-phase Butterworth low-pass filtering, and no feedforward compensation, respectively. Simulated results for the 130 ms reference command are displayed in Figs. 13, 14, and 15. Clearly the best performance is given by the series approach. Figs. 16 and 17 display the reference command after feedforward processing. It is interesting to note that the control efforts are larger for the ZPET with low-pass filtering. In fact, in the ZPET experimental trials, if no additional lowpass filtering was applied, the control efforts were so large that the amplifier was observed to saturate. The reason for this is ZPET squares the gain at all frequencies. Unstable zeros on the right-half complex plane act as high-pass filters. Squaring their gain further amplifies gain at high frequency. Both the 200 ms and 130 ms reference trajectories had sufficient high frequency content to cause amplifier saturation. The series approach, by canceling right-half plane zeros, adds additional low-pass filtering to the feedforward compensator. Consequently, for right-half plane zeros, the series approach is more suitable. For

\[
\begin{align*}
\left\{ \frac{1}{2} \sum_{i=1}^{3} (-1)^{i-1} \left( \frac{1}{2} \right)^{i-1} \left( \frac{5.834}{5.834} \right)^{i-1} \right\} \times \left\{ \frac{1}{2} \sum_{i=1}^{3} (-1)^{i-1} \left( \frac{5.92}{5.92} \right)^{i-1} \right\} \times \left\{ \frac{1}{2} \sum_{i=1}^{8} (-1)^{i-1} \left( \frac{1.407}{1.407} \right)^{i-1} \right\} \\
\left\{ \left( z^2 - 1.9743z + 0.978 \right) \left( z^2 - 0.4886z + 0.646 \right) \left( z^2 + 1.125z + 0.63772 \right) \right\}
\end{align*}
\]  
(29)

\[
\begin{align*}
\left\{ \frac{0.0618(z+0.804)(z-0.7105)(z-0.163429)(z+0.097353)}{0.00033(z+0.804)(z-0.7105)(z-0.163429)(z+0.097353)} \right\} \\
\left\{ \left( z^2 - 1.9743z + 0.978 \right) \left( z^2 - 0.4886z + 0.646 \right) \left( z^2 + 1.125z + 0.63772 \right) \right\}
\end{align*}
\]  
(30)
the more common left-half plane zeros, the ZPET approach is usually sufficient, since tracking at frequencies where these zeros begin to attenuate gain is typically pushing the plant well beyond its inherent capabilities.

The control inputs generated by simulation were subsequently applied experimentally. With reference to Figs. 20 and 23, with no feedforward compensation the overshoots are about 60% for both the 200 ms and 130 ms trajectories. The settling time is on the order of about 1.5 seconds. For the ZPET augmented with a zero-phase lowpass filter, the 200 ms tracking results (Fig. 19) are improved over the no feedforward case. The observed overshoot is still large but reduced to about 25%. The settling time is about one second. There is also a more noticeable reverse reaction of about 0.05 radians. The reverse reaction is typical of when a system model has right-half plane unstable zeros. Also with ZPET applied to right-half plane zeros, the reverse reaction can become more severe. In fact for a single right-half plane zero at $\alpha$, the reverse reaction can be shown to be of maximum magnitude $\frac{1}{1-\alpha}$. When ZPET compensation is applied, the reverse reaction increases and assumes the maximum value $\frac{1}{(1-\alpha)^2}$. This can be verified by evaluating the step responses of $\frac{z^{-1}}{-14(1-\alpha)}$ and $\frac{(z^{-1})(z-\alpha)}{(1-\alpha)^2}$, for $\alpha > 1$. The latter two transfer functions correspond to a simple plant with a right-half plane zero at $\alpha$ and the same plant compensated by ZPET. From Fig. 22, for the 130 ms trajectory applied to the ZPET case, the overshoot is almost 100%. There is clearly a higher mode that is observable and the reverse reaction is about 0.25 radians.

For the series approach, the slow and fast tracking performances yield overshoots of about 15% and 25%, respectively. The rise times are about 250 and 180 milliseconds. The settling time is about 0.75 seconds for the 200 ms trajectory and about one second for the 130 ms trajectory. Also there is very little reverse reaction for the 200 ms reference trajectory. This is expected since the unstable right-half plane zero at 1.407 has been compensated in both phase and gain by the series approximation. For the 130 ms case, the reverse reaction is considerably increased yet still only about 30% that of the ZPET case. At higher speeds, the linearity assumptions become less tenable.

The observed errors in tracking can likely be attributed to plant and model mismatch. The same model was used for the ZPET and series approach, and therefore they were both subject to the same modeling errors. As expected, the tracking errors were larger than that predicted by simulation but the general prediction that the series approach would out perform the ZPET approach was observed. The ZPET approach out-performed the no feedforward approach for the 200 ms reference trajectory. For the high speed 130 ms reference trajectory, however, the no feedforward approach

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**Fig. 13.** Series approach, 130 ms slew.

**Fig. 14.** Simulated ZPET + low pass, 130 ms slew.

**Fig. 15.** No feedforward compensation, 130 ms slew.

**Fig. 16.** Control effort for 200 ms slew, series and ZPET+ low pass.
was better. Canceling the phase and adding the Butterworth filter was not enough. The low-pass filter would have to be retuned with a lower cutoff frequency to realize better performance than the no feedforward case.

More representative models will yield improved performance. The neglect of system nonlinearities in formulating the plant model is not appropriate for high speed tracking. Extensions to the nonlinear case are currently being considered.

V. CONCLUSION

It has been shown that for systems that have right-half plane zeros, the truncated series approach can provide a simple and effective means of compensating the zeros when future reference trajectory information is available. An experimental comparison has been made with no feedforward, ZPET augmented with a zero-phase low-pass filter, and the truncated series approach. The series approach design in this example has been shown to yield the best performance. To compensate zeros on the right-half plane, the series approach is a good choice since by canceling phase as well as gain, it adds low-pass filtering to the feedforward compensation. ZPET adds high-pass filtering since it only cancels phase and squares gain. Though modifying the ZPET controller with low-pass filtering may also yield good performance, it is less direct and somewhat ad hoc. For the more common case of zeros on the left-half plane, ZPET will probably be sufficient. This is because ZPET adds low-pass filtering to the feedforward compensation and the series approach adds high-pass filtering. Also, for the flexible beam example, improved performance can be obtained with the more accurate nonlinear models. Extending the approach described here to nonlinear "nonminimum phase" systems is the subject of future research.
REFERENCES


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