1 Introduction

This document investigates the choice of task space when using a constrained based task specification. The document investigates the implications of a given choice of tasks. It does this by analyzing the behaviour of a mapping between two (scalar) task spaces. The analysis below only considers small deviations. When the derivative of the mapping has a derivative that is not zero and not infinite, the behaviour of the mapped task space for small deviations is the same as the original task space. An explicit formula is derived when the derivative is zero/infinite.

2 Methodology

Consider a constraint whose (scalar) error $e_1$ is controlled by a constrained based task controller to the following behaviour:

$$\frac{de_1}{dt} = K_1 e_1.$$  \hfill (1)

Consider also a second variable $e_2$ that depends on $e_1$ via the non-linear function $f$:

$$e_2 = f(e_1).$$  \hfill (2)

Suppose also that $f(0) = 0$ (in other words, the variable $e_2$ also evolves towards 0). This document investigates by which first order system the behaviour of the mapped task variable $e_2$ can be approximated, and how well it can be approximated:

$$\frac{de_2}{dt} = K_2 e_2 + \epsilon(e_1).$$  \hfill (3)

For small deviations $e_1$, the non-linear function $f$ can be approximated by its Taylor series:

$$f(e_1) = \sum_{n=0}^{N} \frac{1}{n!} f^{(n)}(0) e_1^n + O(e_1^{n+1}).$$  \hfill (4)

In the above approximation, none of the derivatives $f^{(n)}(0)$ should be infinite. The derivative of $f$ towards $e_1$ can also be approximated by a series:

$$\frac{df}{de_1} = \sum_{n=1}^{N} \frac{n}{n!} f^{(n)}(0) e_1^{n-1} + O(e_1^n).$$  \hfill (5)

The chain rule for derivatives can be applied on equation 3, and the derivative of $e_1$ can be substituted using equation 1:

$$K_1 e_1 \frac{df}{de_1} = K_2 f + \epsilon(e_1).$$  \hfill (6)
In equation 3, substituting $e_2$ and its derivative with the Taylor series gives the following expression for $\epsilon(e_1)$:

$$
\epsilon(e_1) = \sum_{n=1}^{N} \frac{K_1 n - K_2}{n!} f^{(n)} (0) e_1^n + O(e_1^{n+1})
$$

(7)

3 Analysis

When the first order derivative of the mapping is not zero and not infinite, and when we choose $K_2 = K_1$, equation 3 is a first order approximation with $\epsilon = O(e_1^2)$. So, in the close neighbourhood of 0, the behaves like an identical first order system.

When the first order derivative of the mapping is zero, and the second order derivative is not zero and not infinite, we can choose $K_2 = 2K_1$ and obtain a better first order approximation with $\epsilon = O(e_1^3)$.

When the first order derivative is infinite, the above method can be applied on the inverse mapping. This leads to a choice $K_2 = K_1/2$ that gives a first order approximation with $\epsilon = O(e_1^2)$.

The above can be easily extended when higher order derivatives are zero or infinite.

4 Case study: the angle between two vectors

Suppose we want to control the angle between to vectors $v_1$ and $v_2$ to a given value $\alpha$. We investigate two options:

- apply the constraint:

$$
e_1 = v_1 \cdot v_2 - \cos(\alpha)
$$

(8)

- or apply the constraint:

$$
e_2 = \arccos(v_1 \cdot v_2) - \alpha
$$

(9)

Suppose that we are controlling $e_1$ and we map the values of $e_1$ on $e_2$ using a non-linear function $f$, we can investigate by which first order system the behaviour of $e_2$ can be approximated (in the neighbourhood of 0). The task spaces of the two constraints are mapped by the following function $f$:

$$
e_2 = f(e_1) = \arccos(e_1 + \cos(\alpha)) - \alpha
$$

(10)

When the deviations for the constraints are small, and $\alpha = \pi/2$, the derivative of $f$ is non-zero and is finite. The first order approximation of the behaviour of the mapped variable is $K_2 = K_1$. This is experimentally verified in figure 1.

When the deviations for the constraints are small, and $\alpha = \pi$, the derivative of $f$ is infinite. The derivative of the inverse function of $f$ is zero; its 2nd derivative is finite and non-zero. The first order approximation of the behaviour of the mapped variable is $K_2 = K_1/2$. This is experimentally verified in figure 2.

5 Conclusion

For large deviations, it is important to consider in which space a constraint is applied. It is the specified constraint variable that will show the behaviour of a first order linear system. For small deviations, e.g. tracking applications, this matters less because, whatever the mapping is, the behaviour will still be as a first order linear system. If the derivative of the mapping is non-zero and finite, the task spaces will even have the same time constant.

The above analysis is closely related to the concept of singularity. Note that the above analysis is also valid for a mapping between joint space and task space. The current analysis only considers mapping between scalar variables.
Figure 1: Behaviour of angle when dot product is controlled, around $\alpha = 90$ deg.

Figure 2: Behaviour of angle when dot product is controlled, around $\alpha = 180$ deg.